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A NOTE ON THE STRUCTURE OF THERMAL CONVECTION IN A SLIGHTLY SLANTED SLOT

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Abstract—We describe experiments conducted with a fluid of high Prandtl number contained in a differentially heated, slightly inclined box of large aspect ratio (width/depth). When the bottom plate is heated and the top plate is cooled convective rolls are superimposed on the basic unicellular Hadley circulation. The structure of this Hadley circulation is calculated in the limit of small inclination angle α, compared with experiment, and shown not to interact with the convective instabilities which occur at a Rayleigh number $Ra > 1708/\cos\alpha$. These convective instabilities take the form of straight longitudinal rolls oriented upslope, so that the complete interior thermal structure of these disturbances is easily measured. Of particular interest is the measurement of mean density gradient reversals for $Ra \gtrsim 4Ra_{\text{crit}}$.

NOMENCLATURE

A, aspect ratio L/D;
α, inclination angle;
D, height of slot;
γ, coefficient of thermal expansion;
g, gravitational acceleration;
κ, thermal diffusivity;
L, width and depth of box;
Nu, Nusselt number;
η, streamfunction for the Hadley circulation;
ψ, streamfunction for the convective rolls;
Pr, Prandtl number $= \nu/\kappa$;
Ra, Rayleigh number;
Ra_{crit}, critical Rayleigh number for onset of convective instability;
T, temperature;
u, upslope velocity;
v, cross-slope velocity;
ν, kinematic viscosity;
w, cross-stream velocity;
x, upslope coordinate;
y, cross-slope coordinate;
z, cross-stream coordinate.

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1. INTRODUCTION

We wish to consider the flow regimes in the 2 dimensional slot shown in Fig. 1. For large Prandtl numbers the governing Boussinesq equations can be put into the form

$$\nabla^2 \boldsymbol{u} - \nabla p + T\cos\alpha\hat{z} - T\sin\alpha\hat{x} = 0 \quad (1.1)$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad (1.2)$$

$$-Ra\boldsymbol{u}\cdot\nabla T = \nabla^2 T \quad (1.3)$$

with Ra = Rayleigh number $= g\gamma\Delta T D^3/\kappa\nu$. All distances have the depth scale D, temperature $-\Delta T$, and velocity $-g\gamma\Delta T D^2/\nu$. In the above g is the gravitational acceleration, κ the thermal diffusivity, ν the kinematic viscosity, γ the coefficient of thermal expansion, and α the tilt angle ($\ll 1$). The boundary conditions are

$$\boldsymbol{u} = 0 \text{ on } z = \pm\tfrac{1}{2}, \quad x = \pm\frac{L}{2D}, \quad y = \pm\frac{L}{2D},$$

$$\nabla T.\hat{n} = 0 \text{ on } x = y = \pm \frac{L}{2D}, \quad (1.4)$$

and

$$T = \pm\tfrac{1}{2} \quad \text{on} \quad z = \pm\tfrac{1}{2},$$

where L is the length = width of the box.

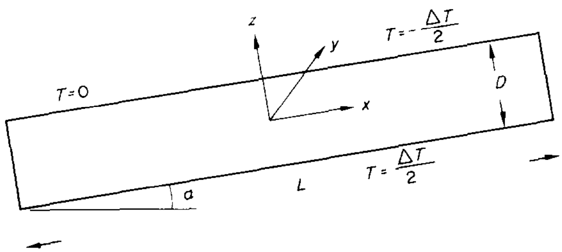

FIG. 1. Geometry and coordinates. The extent of our experimental box in the y direction was L.

2. THE MEAN CIRCULATION

We shall be considering two kinds of flows. A mean steady circulation (MC) which is essentially y independent, and thermal convective instabilities (TC) which are essentially x independent. Now the temperature fluctuations of TC will be totally unaffected by MC if $\partial \mathrm{MC}/\partial x = 0$.* When convective rolls with axes pointed upslope form, there will then be no interaction terms in thermal equation (1.3) between MC and TC. There is an induced upslope velocity due to the second buoyancy term in (1.1) but this will also be x-independent if $\partial \mathrm{MC}/\partial x = 0$ and therefore will not couple into the equations governing TC. The purpose of this section is then to find conditions for which

$$\left.\frac{\partial(\mathrm{MC})}{\partial x}\right|_{x\sim 0} = 0. \quad (2.1)$$

The problem clearly involves the aspect ratio

$$A = \frac{L}{D}. \quad (2.2)$$

* Throughout we are considering motions away from the frictional influence of the side walls. Thus for MC we consider $\partial/\partial y = 0$ and consider that TC is not modified by the sidewalls. For our experimental aspect ratio $L/D \sim 40$ this is certainly alright provided we do not make measurements within a few depths of the sides.

Now if the channel is truly infinite in x, we immediately find an exact solution for MC (subscript m),

$$u_m = \sin\alpha\left(\frac{z^3}{6} - \frac{z}{24}\right) \quad (2.3)$$

and

$$T_m = z \quad (2.4)$$

which is clearly x-independent. We wish then to find out how large A must be to insure that the flow resembles (2.3) and (2.4).

We suppose that MC is independent of y. Then we introduce a streamfunction η such that

$$w = \eta_x$$

and

$$u = -\eta_z.$$

Since α is to be very small we expand

$$T_m(x, z) = T_{m0} + \alpha T_{m1} + \ldots$$

and

$$\eta(x, z) = \eta_0 + \alpha\eta_{m1} + \ldots$$

Then at O(0)

$$\nabla^4\eta_{m0} + T_{m0x} = 0$$

$$-RaJ(\eta_{m0}, T_{m0}) = \nabla^2 T_{m0}.$$

Subject to the boundary conditions we find that we must have

$$T_{m0} = z$$

$$\eta_{m0} \equiv 0.$$

At $0(\alpha)$ the equations are simply

$$\nabla^4\eta_{m1} + T_{m1x} = -1 \quad (2.5)$$

and

$$\nabla^2 T_{m1} + Ra\eta_{m1x} = 0, \quad (2.6)$$

which must be solved subject to

$$\eta_{m1} = \eta_{m1z} = T_{m1} = 0 \quad \text{on} \quad z = \pm\tfrac{1}{2},$$

$$\eta_{m1} = \eta_{m1x} = T_{m1x} = 0 \quad \text{on} \quad x = \pm\frac{A}{2}.$$

Note that for $Ra > 0$ (unstable basic stratification) boundary layer solutions near $x = \pm A/2$ are not possible. For $Ra < 0$ it would appear possible to construct buoyancy layers near $x = \pm A/2$ for large Ra, but this is misleading because temperature actually diffuses beyond the buoyancy layer thickness.

A numerical solution of (2.5) and (2.6) was found by means of the Galerkin method. Up to 25 trial functions in x and up to 10 in z were used to construct η and T. The stream function and temperature deviation are shown for selected cases in Fig. 2. Note that for the unstable cases ($Ra > 0$) the streamlines are parallel to the top and bottom boundaries throughout most of the channel, and nonzero T_{m1} values occur only near the ends if $A = 10$.

If $Ra > 0$ and $A \lesssim 5$ or if $Ra < 0$ the circulation near $x = 0$ is definitely affected by the end walls. In a short box with $Ra > 0$ the release of potential energy near the ends accelerates the flow beyond its normal (equation (2.3)) magnitude. If $Ra < 0$, the basic stability inhibits vertical velocities and retards the circulation. The vertical velocities at the channel ends cause temperature variations which diffuse into

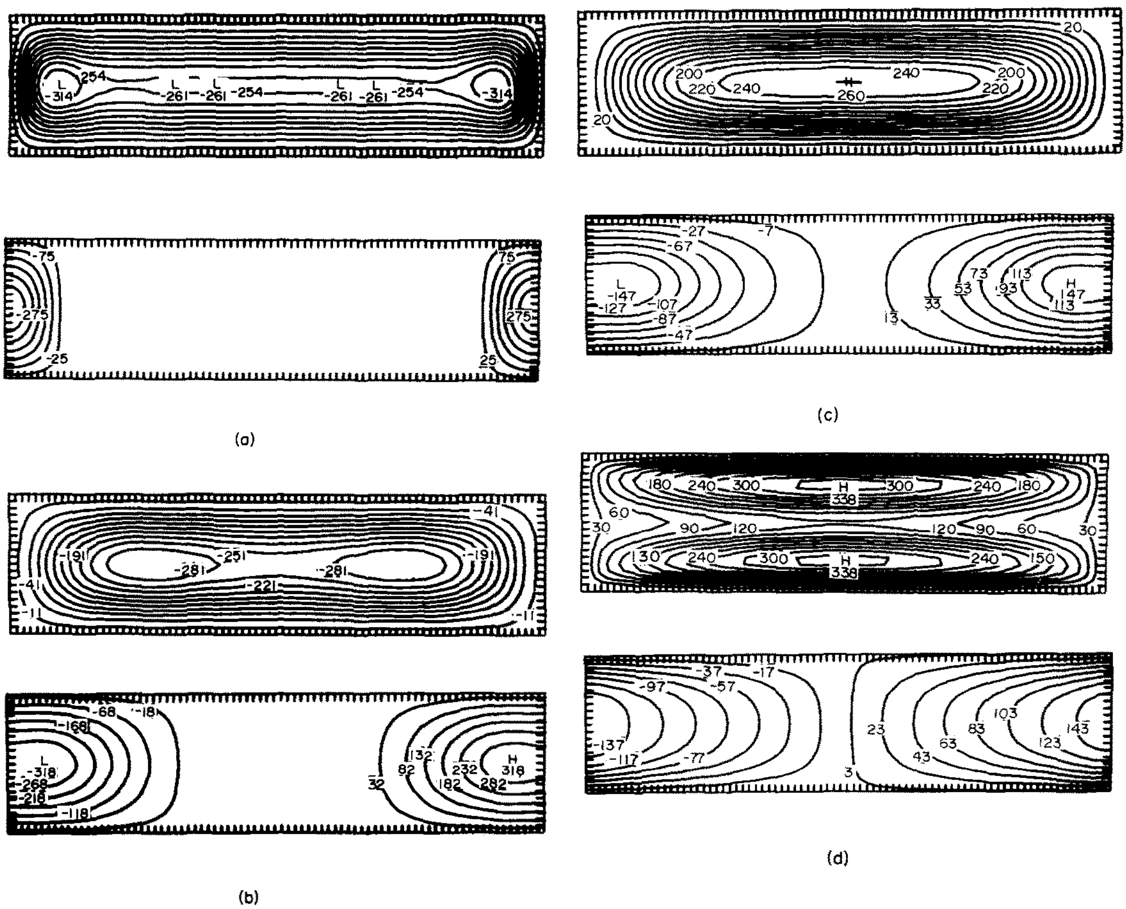

FIG. 2. Streamlines (upper) and isotherms (lower) for the corrections. In (a) $A = 10$, $Ra = 1700$, in (b) $A = 3$, $Ra = 1700$, in (c) $A = 3$, $Ra = -10^3$, and in (d) $A = 3$, $Ra = -5 \times 10^6$.

the interior. Thus a characteristic of the solutions for $Ra < 0$ is the development of an interior temperature distribution which is approximately linear with x at $z = 0$. Figures 3 and 4 show the development of these gradients along with some data taken by traversing a fine thermocouple probe through a high Pr silicon oil at $y = z = 0$. Figure 5 shows how the temperature T_{m1} varies with z near the ends. In spite of the fact that the perspex sidewalls are not perfect insulators, the agreement between this simple theory and experiment is rather good.

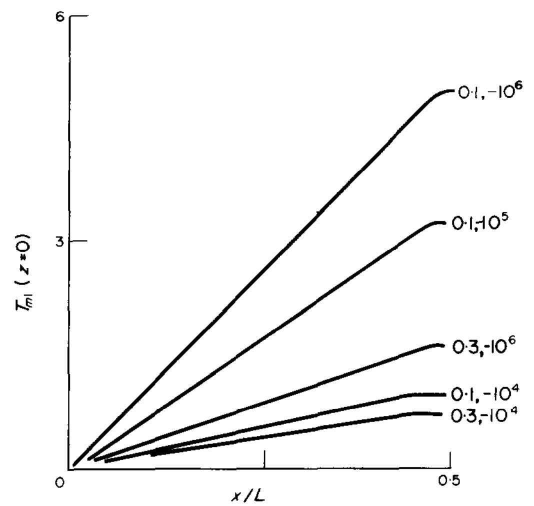

FIG. 3. Theoretical horizontal temperature profiles at $z = 0$ for the values of (A^{-1}, Ra) shown.

In conclusion, we have shown that if $A \gtrsim 10$ the interior for $Ra > 0$ is identical to that given for infinite plates in (2.3) and (2.4). In particular $\eta_{m1x} \sim T_{m1x} \sim 0$ so no interaction with the upslope rolls to be described in the next section will occur. If $Ra < 0$ strong blocking affects the interior flows and the velocity and temperature fields are significantly modified from (2.3) and (2.4). Linear stability analyses of these small α flows based on (2.3) and (2.4) (Liang and Acrivos [7], Hart [4], Birikh *et al.* [1]) should be applied with caution if

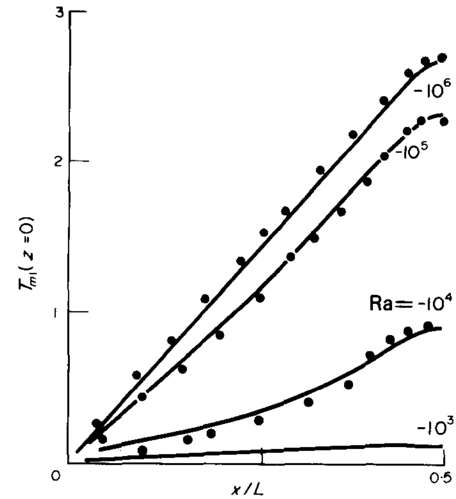

FIG. 4. Horizontal temperature profiles (solid) compared with experiment. $A = 5$.

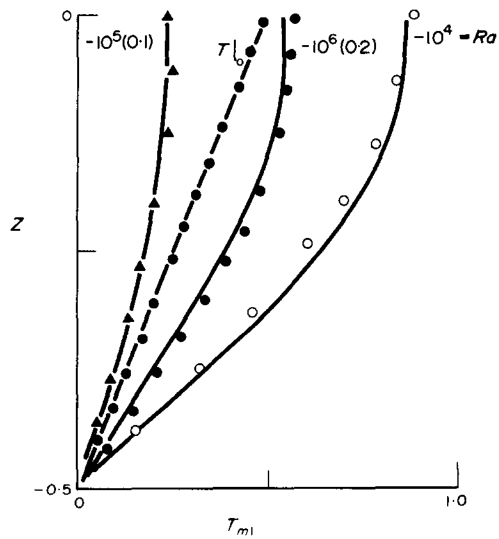

FIG. 5. Comparison of theoretical vertical temperature distributions (solid) with experiment at $x/L = 0{\cdot}47$ and at $x = 0$. $A = 5$ and scale factors for the plots are shown in parenthesis.

$Ra > 0$ and $A \lesssim 5$ or if $Ra < 0$ and $-ARa \gtrsim 2000$. Using (2.6) and applying the boundary conditions it is easily seen that the mean circulation Nusselt number is

$$Nu = 1 + O(\alpha^2).$$

Symmetry arguments in fact require that Nu be a function of even powers of α only.

3. UNSTABLE FLOWS AND THE STRUCTURE OF 2-DIMENSIONAL THERMAL CONVECTION

If Ra is greater than $1708/\cos\alpha$ we expect thermal instabilities to occur. Previous stability theories based on (2.3) and (2.4) have shown that in situations like this one expects upslope rolls to occur (Liang and Acrivos [7], Kurtweg [6], Hart [4]). Thus, using the results of the previous section with $A \sim 40$ we can take $T = T_{m0} + T'(y, z)$. If we now construct a convective roll stream function ψ (where $w = \psi_y$ and $v = -\psi_z$), we find

$$\nabla^4\psi + T'_y = 0 \tag{3.1}$$

$$-RaJ(\psi, T') = \nabla^2 T' + Ra\psi_y. \tag{3.2}$$

Subject to boundary conditions (1.4) and neglecting the effects of the sidewalls on observations made in the center of our $A \sim 40$ apparatus it is seen that (3.1) and (3.2) are the equations for 2-dimensional thermal convection of infinite Prandtl number which have been studied numerically by a number of authors.

The apparatus used in the present experiments takes advantage of the particular orientation of the convection. A square convection tank with $\frac{1}{2}$ in. thick, precision ground aluminum boundaries at $z = \pm\frac{1}{2}$ and sidewalls is elevated at $x = +A/2$ so that the fluid layer is tipped up 2° from horizontal. A $\frac{3}{64}$ in. gap is cut along $x = A/6$ in the top plate which conducts an L-shaped 0·003 in. thermocouple probe, extending 2 cm downslope from the crack, into the working fluid. An automated drive positions the probe between $z = -\frac{1}{2}$ and $z = 0{\cdot}4$ and $y = \pm 3A/8$. The working fluid was 100 cs silicone oil ($Pr \sim 900$). ΔT was slowly increased over a period of 7 weeks, with measurements being made every few days. In this manner about 8 sets of data on $T'(y, z)$ were obtained for several Rayleigh numbers. Figure 6 shows the results for some typical cases. One can clearly see the development of higher harmonics

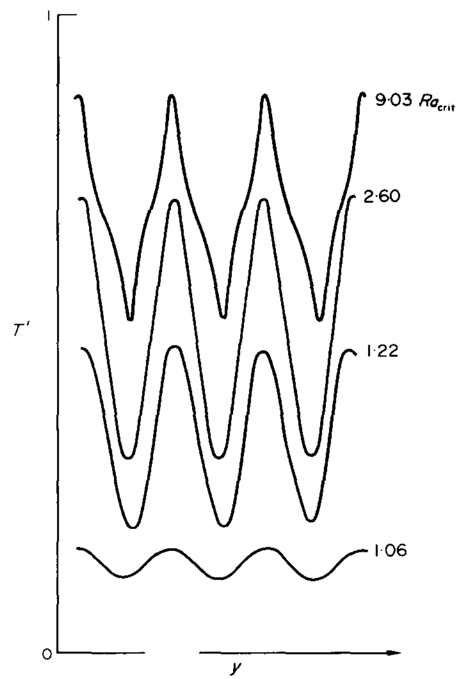

FIG. 6. Sample data on the temperature field at $z = 0$ for the values of Ra shown.

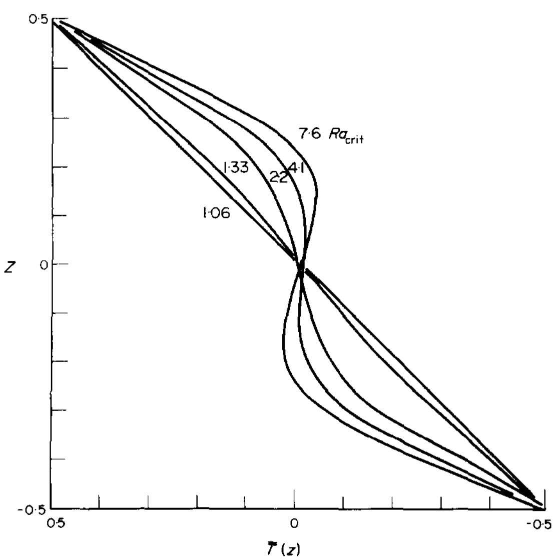

FIG. 7. Horizontally averaged temperature distributions (the average was taken over one roll).

in the temperature field as Ra increases. The most interesting fact coming from these measurements is the existence of mean density gradient reversals for $Ra \gtrsim 4Ra_{crit}$. Figure 7 shows $\bar{T}(z)$ for several Ra. These observations at essentially infinite Pr complement similar findings by Gille [3] in an air layer, except here since we know the form of the disturbances from direct measurements we can say that the reversal is a characteristic of each steady laminar roll. This finding also agrees with the numerical calculations of Fromm [2] and Herring [5]. The reversal reaches a maximum at about $6Ra_{crit}$. That the flow starts to develop strong higher harmonics at about the same time the density gradient reverses appears to be coincidental since such reversals were observed in the one wave mean field calculations of Herring [5].

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NOTE SUR LA STRUCTURE DE LA CONVECTION THERMIQUE DANS UNE FENTE LEGEREMENT INCLINEE

Résumé—On décrit des expériences menées sur un fluide à grand nombre de Prandtl renfermé dans une boite légèrement inclinée, différentiellement chauffée et à grand rapport de forme (largeur/profondeur). Quand la plaque inférieure est chauffée et la plaque supérieure refroidie, des rouleaux de convection sont superposés à la circulation fondamentale unicellulaire de Hadley. La structure de cette circulation de Hadley est calculée dans la limite d'un angle d'inclinaison α petit, comparée à l'expérience et on montre qu'elle n'agit pas sur les instabilités de convection qui apparaissent pour un nombre de Rayleigh $R_a >$ 1708/cos α. Ces instabilités de convection prennent la forme de rouleaux longitudinaux droits orientés vers le haut de la pente si bien que la structure thermique intérieure complète de ces perturbations est facilement mesurée. La mesure des inversions du gradient de densité moyen pour $R_a \geqslant 4\, R_{a\,crit}$ est d'un intérêt particulier.

EIN BEITRAG ZUR STRUKTUR DER THERMISCHEN KONVEKTION IN EINEM SCHWACH GENEIGTEN SPALT

Zusammenfassung—Es werden Versuche beschrieben mit einem Fluid mit hoher Prandtl-Zahl, das sich in einem veränderlich beheizten, leicht geneigten Behälter mit einem grossen Höhen-Breiten-Verhältnis befindet. Wenn die Bodenplatte beheizt und die Deckplatte gekühlt wird, überlagern sich Konvektionswalzen der ursprünglich einzelligen Hadley-Zirkulation. Die Struktur dieser Hadley-Zirkulation wird im Bereich kleiner Neigungswinkel α berechnet und mit dem Experiment verglichen. Dabei zeigt sich, dass sie nicht mit den konvektiven Instabilitäten zusammenwirkt, die bei Rayleigh-Zahlen $Ra >$ 1708/cos α auftreten. Diese konvektiven Instabilitäten haben die Form gerader, aufwärtsgerichteter Längswalzen, so dass der gesamte innere thermische Aufbau dieser Störungen leicht gemessen werden kann. Von besonderem Interesse ist die Messung des Umschlags des mittleren Dichtegradienten für $Ra > 4\, Ra_{krit.}$.

О СТРУКТУРЕ ТЕПЛОВОЙ КОНВЕКЦИИ В ЩЕЛИ С НЕБОЛЬШИМ НАКЛОНОМ

Аннотация—Описываются опыты по конвекции в жидкости с большим числом Прандтля, заполняющей дифференциально нагретую слегка наклоненную полость с болбшим отношением ширины к высоте. При нагреве нижней и охлаждении верхней пластины конвективные валы накладываются на основную одноячеистую циркуляцию Хедли. Структура основной циркуляции Хедли. Структура основной циркуляции Хедли рассчитывается в пределе малого по сравнению с экспериментом угла наклона α. Показано, что эта структура не взаимодействует с конвективными неустойчивостями, которые возникают при значениях числа Релея $R_a > 1708/\cos\alpha$. Конвективная неустойчивость приводит к появлению прямых продольных валов, располагающихся вверх по наклону, поэтому измерение внутренней тепловой структуры этих возмущений не представляет больших трудностей. Особый интерес представляет измерение изменений направления среднего градиента плотности для $R_a \gtrsim 4R_a$ крит.